Equivalence of ferromagnetic spin models on trees and random graphs

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Abstract. Recently we have found identical behaviour for various spin models on 'thin' random graphs—Feynman diagrams—and the corresponding Bethe lattices. In this paper we observe that the ratios of the saddle-point equations in the random graph approach are identical to the fixed point(s) of the recursion relations which are used to solve the models on the Bethe lattice. The loops in the random graphs thus have no influence on the thermodynamic limit for such ferromagnetic spin models.

We consider the correspondence explicitly for Ising and q state Potts models and also note that multispin interaction models on cacti admit a similar correspondence with a randomized version of the cacti graphs which contain loops.

1. Introduction

In a series of papers [1] we have found identical behaviour for various spin models on 'thin' random graphs—Feynman diagrams—and the corresponding Bethe lattices. The key observation for the approach of [1] was made in [2], where it was noted that spin models on random graphs could be treated as the $N \to 1$ limit of the appropriate Hermitian matrix model†. The $N \to \infty$ limit in such models is already familiar from the theory of two-dimensional (2D) gravity, where it generates planar random graphs and the expansion in $1/N$ is an expansion in the topology of the graphs. The $N \to 1$ limit weights all topologies equally so the graph/surface correspondence of the $N \to \infty$ limit, which is important for 2D gravity and string theory applications, is lost. Nonetheless, the generic random graphs of [1] (which we denoted as 'thin' graphs to distinguish them from the 'fat' graphs of the planar limit) are still of interest in a statistical mechanical context because their locally tree-like structure [4] means that one obtains mean-field transitions for spin models which live on them. They offer one great advantage over genuine tree-like structures such as the Bethe [5] lattice in that they have no boundary, so one is not forced to discard the majority of vertices in a simulation or be subject to the gymnastics of considering only points 'deep within' the lattice analytically.

One remarkable fact that has emerged from all of the simulations and calculations in [1] is that the thermodynamic behaviour of spin models with ferromagnetic couplings on the Feynman diagrams is *identical* to that on the corresponding Bethe lattice, even down to non-universal features such as the critical temperature. The loops do play a role when

† Analogous methods have also been independently derived starting from a probabilistic viewpoint by Whittle in a series of papers [3].

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one considers antiferromagnetic couplings, which offered the possibility of frustration and the presence of spin glass rather than antiferromagnetic ordering at low temperatures. The similarity was all the more interesting because the methods of solution for the two classes of model are, at first sight, completely different. In the random graph models one looks at a saddle-point equation for the 'fields' of the model and the saddle-point action, which is exact in the $n = \sharp\text{vertices} \to \infty$ limit, gives the free energy. All the thermodynamic quantities such as the magnetization, energy, specific heat and susceptibilities may then be calculated from this. For the Bethe lattice models on the other hand, the hierarchical structure of the lattice allows one to construct a recursion relation for partition functions on subtrees that is iterated shell by shell to get the behaviour at the central point (i.e. 'deep within' the lattice). The asymptotic behaviour of this iteration (fixed points, limit cycle, chaotic, etc) then indicates the phase structure of the model [6, 8] and provides a fruitful link with the theory of dynamical systems, particularly for multispin interaction and frustrated models.

In this paper we show that the previous heuristically observed identity between the Bethe lattice and its random graph equivalent is explained by the fact that the ratio of the saddle-point equations in the random graph model gives exactly the fixed point(s) of the recursion relation in the Bethe lattice models. The statement that we have made, and repeated in many of [1] that the loops are irrelevant for ferromagnetic phase transitions on random graphs is thus put on a firmer footing. In essence, the content of the saddle-point equations for the random graph models is identical to the recursion relations of the Bethe lattice, which of course contains no loops by definition. We demonstrate this explicitly for the Ising and Potts models and, in order to show that the result is not confined to nearest-neighbour interactions and the Bethe lattice proper, we also consider a three-site interaction model [6, 7, 9, 10] on a cactus graph and its randomized equivalent and find exactly the same equivalence. For simplicity, we shall consider three-regular (ϕ^3) random graphs and cacti built from triangles throughout this paper, which correspond to a Bethe lattice with three neighbours for each site, but the results we discuss hold for any number of neighbours.

2. The Ising model

The partition function for the Ising model with Hamiltonian

$$H = \beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{1}$$

($\sigma_{i,j} = \pm 1$) on three regular random graphs† with $2n$ vertices may be written as [1, 2]

$$Z_n(\beta) \times N_n = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{2n+1}} \int \frac{d\phi_+ d\phi_-}{2\pi \sqrt{\det K}} \exp(-A) \tag{2}$$

where N_n is the number of undecorated (no spin) ϕ^3 graphs with $2n$ vertices

$$N_n = \left(\frac{1}{6}\right)^{2n} \frac{(6n-1)!!}{(2n)!!}. \tag{3}$$

The action is

$$A = \frac{1}{2} \sum_{a,b} \phi_a K_{ab}^{-1} \phi_b - \frac{\lambda}{3}(\phi_+^3 + \phi_-^3) \tag{4}$$

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† In principle the thin graph partition functions are for an annealed ensemble of all graphs with $2n$ vertices. In practice self-averaging appears to ensure that one can consider a *single* random graph, at least for ferromagnetic spin models such as those discussed here.

where $a, b = \pm$ and the propagator is

$$K_{ab} = \begin{pmatrix} \sqrt{g} & \frac{1}{\sqrt{g}} \\ \frac{1}{\sqrt{g}} & \sqrt{g} \end{pmatrix} \tag{5}$$

where $g = \exp(2\beta)$ relates the coupling in the action to the inverse temperature β.

The saddle-point equations $\partial A / \partial \phi_\pm = 0$ may be written, after scaling out λ, as

$$\phi_+ = \sqrt{g}\phi_+^2 + \frac{1}{\sqrt{g}}\phi_-^2$$

$$\phi_- = \sqrt{g}\phi_-^2 + \frac{1}{\sqrt{g}}\phi_+^2 \tag{6}$$

and one finds the solutions

$$\phi_+, \phi_- = \frac{(g-1)}{g}$$

$$\phi_+, \phi_- = \frac{1 + g \pm \sqrt{(g+1)(g-3)}}{2g}. \tag{7}$$

The upper high-temperature paramagnetic solution bifurcates to the low-temperature ferromagnetic solution in a mean-field transition at $g = 3$, as can be seen by looking at the magnetization order parameter, which in this notation is

$$M = \frac{\phi_+^3 - \phi_-^3}{\phi_+^3 + \phi_-^3}. \tag{8}$$

At first sight these solutions, and the method by which they were obtained, bear little relation to the Bethe lattice calculation [6] where the partition function may be expressed in a recursive manner as

$$Z = \sum_{\sigma_0} \exp(h\sigma_0)[g_n(\sigma_0)]^3 \tag{9}$$

where the central spin is σ_0, the external field is h, we have n 'shells' in the Bethe lattice and

$$g_n(\sigma_0) = \sum_s Q_n(\sigma_0|s) \tag{10}$$

with

$$Q_n(\sigma_0|s) = \exp(\beta\sigma_0 s_1 + hs_1) \prod_{j=1}^{2} Q_{n-1}(s_1|t^{(j)}) \tag{11}$$

reflecting the decomposition of the tree into trunk and branches. s denotes all the spins apart from σ_0 on the jth subtree and $t^{(j)}$ denotes the spins apart from s_1 on the jth branch of the subtree. Letting

$$x_n = \frac{g_n(-)}{g_n(+)} \tag{12}$$

and setting the external field h to zero, we finally find

$$x_{n+1} = \frac{\frac{1}{\sqrt{g}} + \sqrt{g}x_n^2}{\sqrt{g} + \frac{1}{\sqrt{g}}x_n^2}. \tag{13}$$

This recursion relation gives a single fixed point in the paramagnetic phase ($g < 3$) and a pair of stable fixed points with one unstable fixed point in the ferromagnetic phase ($g > 3$).

Although the phase-transition points are identical, they have apparently appeared by very different paths on the Bethe lattice and the random graphs. However, a much closer similarity appears between equations (6) and (13) when one divides the saddle-point equation for ϕ_- by that for ϕ_+ to get

$$\frac{\phi_-}{\phi_+} = \frac{\sqrt{g}\phi_-^2 + \frac{1}{\sqrt{g}}\phi_+^2}{\sqrt{g}\phi_+^2 + \frac{1}{\sqrt{g}}\phi_-^2} \tag{14}$$

which is identical to equation (13) when we set $x_n = \phi_-/\phi_+$ and go to the fixed point(s). The loops in the random graphs have thus made no contribution to the saddle-point equations, which are completely equivalent to the fixed-point solutions of equation (13).

Although we have only detailed here the results when the external field is zero, the divided saddle-point equations are still identical to the Bethe lattice recursion relation at the fixed point in non-zero field when the identification $x = \phi_-/\phi_+$ is made. As a consequence, quantities such as the magnetization are also given by identical formulae on thin graphs and on the Bethe lattice (for a site 'deep within' the lattice). The Ising model is the simplest decoration that we could consider for our Bethe lattice and ϕ^3 random graphs, but the correspondence between the critical behaviour on the two classes of lattices is no fluke and is maintained for other spin models, as we shall see in sections 3 and 4.

3. The Potts model

The q state Potts model Hamiltonian may be taken to be

$$H = \beta \sum_{\langle ij \rangle} (\delta_{\sigma_i,\sigma_j} - 1) \tag{15}$$

where the spins σ_i now take on q different values. The general formula for the partition function is similar to the Ising model, but there are now q fields ϕ and the action is given by

$$A = \frac{1}{2} \sum_{i=1}^{q} \phi_i^2 - c \sum_{i<j} \phi_i \phi_j - \frac{\lambda}{3} \sum_{i=1}^{q} \phi_i^3 \tag{16}$$

where the coupling is related to the inverse temperature by

$$c = \frac{1}{(\exp(2\beta) + q - 2)}. \tag{17}$$

One finds the low-temperature saddle-point solutions

$$\begin{aligned}
\phi_{1\ldots q-1} &= \frac{1 - (q-3)c - \sqrt{1 - 2(q-1)c + (q-5)(q-1)c^2}}{2} \\
\phi_q &= \frac{1 + (q-1)c + \sqrt{1 - 2(q-1)c + (q-5)(q-1)c^2}}{2}
\end{aligned} \tag{18}$$

(and another low-temperature branch with the signs in front of the square roots reversed) which can be obtained self-consistently by imposing the observed symmetry-breaking pattern on the full action in equation (16) to get

$$A = \tfrac{1}{2}(q-1)[1 - c(q-2)]\phi^2 - \tfrac{1}{3}(q-1)\phi^3 + \tfrac{1}{2}\tilde{\phi}^2 - \tfrac{1}{3}\tilde{\phi}^3 - c(q-1)\phi\tilde{\phi} \tag{19}$$

with $\phi = \phi_{1\ldots q-1}$, $\tilde{\phi} = \phi_q$ and λ scaled out. Similarly, at high temperatures the saddle-point solution is $\phi_{1\ldots q} = \phi_0 = 1 - (q-1)c$ which can be obtained from the effective action

$$A_0 = \frac{q}{2}(1 - c(q-1))\phi_0^2 - \frac{q}{3}\phi_0^3. \tag{20}$$

Although we started with q fields, the final saddle-point solution only requires two fields in its solution as a consequence of the symmetry-breaking pattern in the low-temperature phase, just as for the Ising model.

The ratio of the saddle-point equations following from equation (19) may be written as

$$\frac{\phi}{\tilde{\phi}} = \frac{\phi^2 + c\tilde{\phi}^2}{c(q-1)\phi^2 + (1 - c(q-2))\tilde{\phi}^2} \tag{21}$$

and it is not immediately apparent that this is equal to the recursion relations for the Potts model on a Bethe lattice in [11], which may be written in the form

$$x_{n+1} = \left[\frac{1 + x_n/c}{\exp(2\beta) + (q-1)x_n}\right]^2. \tag{22}$$

However, one can rearrange equation (21) into

$$z = \frac{1 + z^2/c}{(q-1)z^2 + \exp(2\beta)} \tag{23}$$

where $z = \phi/\tilde{\phi}$. If we now make the identification $z^2 = x$ we recover the fixed point of the recursion relation in equation (22). We have thus found, just as for the Ising model (which is, after all, a $q = 2$ state Potts model) that the fixed points of the Bethe lattice recursion relations and the saddle-point equations on thin graphs are identical.

4. Multisite interaction models and loopy cacti

A Husimi tree (or 'cactus' graph) is a tree-like structure rather like the Bethe lattice, but the individual components are polygons joined at their vertices, rather than simply vertices and links. It is characterized by the number of polygons articulated at each vertex, γ. A cactus with $\gamma = 2$ is shown in figure 1 along with the Bethe lattice for which it forms a sort of medial graph. For $\gamma > 2$ one does not have such a direct correspondence but the hierarchical structure still, of course, remains which allows one to deploy similar techniques to the Bethe lattice solution, using recursion relations that operate shell by shell. Since the individual units are now more complicated than vertices, it is possible to put models with multisite interactions on such trees and investigate their phase behaviour [8–10]. Writing down a thin graph-style action for such models corresponds to allowing (predominantly large) loops in the branches of the cacti, in the same manner as the Feynman diagram expansion introduces large loops into a locally tree-like structure. Given the results of the previous sections relating the critical behaviour on the Bethe lattice and thin graphs one might expect that the loopy cacti would display exactly the same behaviour as the loop-less cacti.

As an example we take the following multispin interaction model (a special case of the model solved by Wu and Wu [7] on the Kagomé lattice)

$$H = \beta\left(J_3 \sum_{\triangle} \sigma_i\sigma_j\sigma_k + h \sum_i \sigma_i\right) \tag{24}$$

where the three-spin sum is over triangles $\triangle$. If one considers the possible spin configurations on the triangle of the Husimi tree, namely $(+++), (---), (++-), (+--)$, the appropriate action can be written for the lattice which looks locally like figure 1 (but has large loops of triangles) by thinking of the spins as residing on the links of the underlying
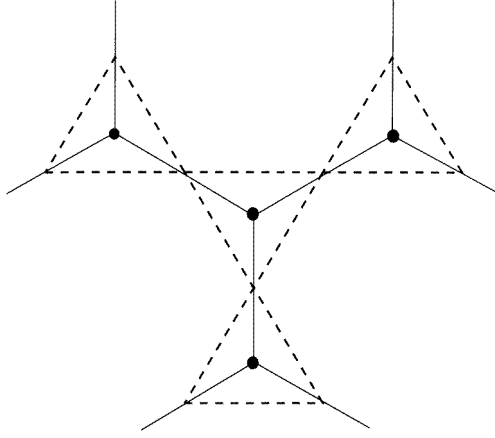
Figure 1. A section of a Husimi tree (broken) composed of triangles with $\gamma = 2$. The underlying Bethe lattice (full) is also shown, along with the 'ghost' nodes at the centre of each triangle.

ϕ^3 graph. One then has 'ghost' links joining 'ghost' vertices at the centre of each of the triangles, which gives

$$A = \frac{\mu}{2}\phi_+^2 + \frac{1}{2}\phi_-^2 - \mu^2\phi_+^2\phi_- - z\mu\phi_-^2\phi_+ - \frac{z\mu^3}{3}\phi_+^3 - \frac{1}{3}\phi_-^3 \tag{25}$$

where $\mu = \exp(2\beta h)$, $z = \exp(2\beta J_3)$ and we have again pre-emptively scaled out the vertex counting factor λ for simplicity. Note that an edge on which a positive spin resides, represented by the 'propagator' for ϕ_+, picks up a factor of $1/\mu$ to avoid double counting the external field contribution where two triangles join. If we now consider the saddle-point equations for the model

$$\begin{aligned}
\phi_+ &= z\mu^2\phi_+^2 + 2\mu\phi_+\phi_- + z\phi_-^2 \\
\phi_- &= \mu^2\phi_+^2 + 2z\mu\phi_+\phi_- + \phi_-^2
\end{aligned} \tag{26}$$

we find that their ratio is again exactly the recursion relation derived for the model on a loop-less cactus (Husimi tree) in [10].

The Hamiltonian may be extended to include nearest-neighbour interactions as well

$$H = \beta\left(J_3 \sum_\Delta \sigma_i\sigma_j\sigma_k + J_2 \sum_{ij} \sigma_i\sigma_j + h \sum_i \sigma_i\right) \tag{27}$$

and similar reasoning to the pure three-spin interaction case above leads to the action

$$A = \nu\left(\frac{\mu}{2}\phi_+^2 + \frac{1}{2}\phi_-^2 - \mu^2\phi_+^2\phi_- - z\mu\phi_-^2\phi_+ - \frac{z\nu^2\mu^3}{3}\phi_+^3 - \frac{\nu^2}{3}\phi_-^3\right) \tag{28}$$

for the model on loopy cacti, where the additional parameter $\nu = \exp(2\beta J_2)$. The overall factor of ν is irrelevant for solving the saddle-point equations and we find that their ratio is identical to the recursion relation presented for the model on a loop- less cactus in [9]

$$\frac{\phi_+}{\phi_-} = \frac{z\mu^2\nu^2\phi_+^2 + 2\mu\phi_+\phi_- + z\phi_-^2}{\mu^2\phi_+^2 + 2z\mu\phi_+\phi_- + \nu^2\phi_-^2}. \tag{29}$$

The loops in the cacti have thus not altered the critical behaviour of the models from the loop-less case, at least for ferromagnetic couplings, where one is looking for fixed points of the recursion relations rather than more exotic behaviour.

5. Discussion

We have seen that for all the models considered—Ising, Potts, multisite interaction, and mixed interaction—the randomization of the Bethe lattice or Husimi tree by the introduction of loops has no effect on the thermodynamic limit for ferromagnetic phase transitions. This irrelevance of loops is also to be expected on graph theoretic grounds [4], for it is known that the loop distribution on a regular random graph is very strongly weighted to large loops†. This provides a justification for calling the thin graphs locally tree-like.

The dynamical systems aspects of the recursive approach on Bethe lattices and Husimi trees are lost in the ratios of the saddle-point equations when one moves to randomized structures, so phases with staggered order (in particular, antiferromagnetic order) which appear as cycles in the solution of the recursion relations are also absent. This is intuitively reasonable, because the ϕ^3 graphs contain both odd and even loops, so the shell-by-shell alternation of spin values that characterizes antiferromagnetic order on the Bethe lattice proper would lead to frustration via the loops.

Indeed, there is strong evidence [1, 2] both analytical and numerical that the Ising model with purely antiferromagnetic couplings displays a spin glass phase at sufficiently low temperature on ϕ^3 graphs rather than the antiferromagnetic phase of the Bethe lattice. It would be extremely interesting to investigate some of the more esoteric dynamical phenomena in the recursion relations of the multisite interaction models with antiferromagnetic couplings that are highlighted in [9, 10] in order to see what, if any, phenomena they corresponded to in the randomized/loopy case.

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† More precisely, on three regular graphs the number of i-cycles, that is loops with i edges, is asymptotically a Poisson random variable with mean $2^i/(2i)$.

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